Eur. Phys. J. C 54, 685–690 (2008)
DOI 10.1140/epjc/s10052-008-0560-3

THE EUROPEAN
PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

On the spectra of noncommutative 2D harmonic oscillator

Jian Jing[1,a], Shi-Hua Zhao[2], Jian-Feng Chen[1], Zheng-Wen Long[3]

[1] Key Lab for Nanomaterials, Ministry of Education, Beijing University of Chemical Technology, Beijing 100029, P.R. China
[2] School of Basic Science & Arts, China University of Mining and Technology (Beijing), Beijing 100083, P.R. China
[3] Laboratory for Photoelectric Technology and Application, Department of Physics, GuiZhou University, GuiYang 550025, P.R. China

Received: 19 September 2007 /
Published online: 8 March 2008 –

Abstract. The spectra and wave functions of the 2-dimensional harmonic oscillator in a noncommutative plane are revised by using the path integral formulation in coordinate space and momentum space, respectively. We perform the path integral formulation in coordinate space first. Then we study this problem in momentum space. The propagator is computed both in coordinate space and in momentum space. The modification due to noncommutativity of eigenvalues and eigenfunctions is studied. Both the small and large noncommutative parameter limits are discussed.

PACS. 11.10.Ef

1 Introduction

In the past few years, noncommutativity has attracted much attention owing to the development of string theory [1–3]. It is widely believed that the open string's end points will be noncommutative in the presence of a background NS-NS B-field. This shows that the coordinates of D-branes, which the open strings are attached to, are noncommutative [4–7]. The fluctuations of the brane are described by noncommutative field theories. As a result, there is a considerable number of papers dealing with field theories on noncommutative space. It has been proven in [8, 9] that some non-trivial phenomena would occur in perturbative quantum field theories. The non-perturbative aspect of field theories on noncommutative space has also been studied extensively since the work of [10].

The one particle sector of noncommutative quantum field theories, i.e., noncommutative quantum mechanics (NCQM), has also been studied in various respects [11–14]. The eigenvalue problem of NCQM is the focus one [15–18]. In most of the works, the authors map the noncommutative quantum space to a commutative one and then use the operator form to study these problems. However, the problem of operator ordering may arise. The other form of quantum mechanics, say, the path integral formulation, especially analyzing eigenvalues and eigenfunctions, has received less attention. The advantage of the path integral formulation is that all the numbers are C-numbers so the problem of operator ordering can be avoided. The present paper is devoted to filling of this gap.

The key point of the path integral is the construction of the propagator. Both eigenvalues and wave functions can be read from it once the propagator is constructed. We shall reexamine the spectra and wave functions of the noncommutative 2-dimensional harmonic oscillator from the path integral point of view. Although we only analyze a simple model, we hope that this paper may give some hints to analogous problems or serve pedagogical purposes at least. The organization of our paper is as follows: in Sect. 2 we shall start from the classical noncommutative 2-dimensional harmonic oscillator and then map it to the ordinary classical plane. The propagator is constructed, and then both eigenvalues and wave functions are obtained. In Sect. 3, we shall perform the path integral directly in momentum space. Some further discussion will be given in Sect. 4.

2 Path integral in coordinate space

The noncommutative quantum plane is defined by

$$[\hat{X}^i, \hat{X}^j] = \mathrm{i}\theta^{ij}\,, \quad i, j = 1, 2\,, \tag{1}$$

where θ^{ij} is the noncommutative parameter, an antisymmetric tensor of dimension of (length)^2. Since we only concentrate on the 2-dimensional case in this paper, the noncommutative parameter θ^{ij} can be chosen as $\theta^{ij} = \theta\epsilon^{ij}$. The full algebras of NCQM are characterized by the algebras (1) together with the following commutative relations [11–14]:

$$[\hat{X}^i, \hat{P}_j] = \mathrm{i}\hbar\delta^i_j\,, \quad [\hat{P}_i, \hat{P}_j] = 0\,. \tag{2}$$

[a] e-mail: jingjian@mail.buct.edu.cn

The commutation relations (1) imply that there is another Heisenberg-type relation besides the standard one in the noncommutative plane,

$$\Delta X^1 \Delta X^2 \sim \theta \,. \tag{3}$$

It means that one cannot localize a particle's position exactly. As a result, the wave functions in the noncommutative coordinate space $\psi(X^i,t)=\langle X^i,t|\psi\rangle$ lose their exact meaning. However, one can map the algebras of NCQM, (1) and (2), to the ordinary ones, which are characterized by

$$\begin{aligned} &[\hat{x}^i,\hat{x}^j]=0\,, \quad [\hat{x}^i,\hat{p}_j]=\mathrm{i}\hbar\delta^i_j\,,\\ &[\hat{p}_i,\hat{p}_j]=0\,, \end{aligned} \tag{4}$$

via the following transformation:

$$\hat{X}^i=\hat{x}^i-\frac{1}{2\hbar}\theta^{ij}\hat{p}_j\,, \quad \hat{P}_i=\hat{p}_i\,. \tag{5}$$

Accordingly, the Hamiltonian in the noncommutative space $H(\hat{X}^i,\hat{P}_i)$ should be replaced by the Hamiltonian in ordinary space, $H(\hat{X}^i,\hat{P}_i)\to H(\hat{x}^i,\hat{p}_i)$, with

$$H(\hat{X}^i,\hat{P}_i)=H(\hat{x}^i-\frac{1}{2\hbar}\theta^{ij}\hat{p}_j\,, \quad \hat{p}_i)\,. \tag{6}$$

After the standard substitutions $\hat{x}^i\to x^i, \hat{p}_i=\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^i}$, and $\hat{H}(\hat{x}^i,\hat{p}_i)\to H\left(x^i,\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^i}\right)$ (the operator ordering problem may arise at this stage), one can write down the Schroedinger equation and give the wave functions $\psi(x^i,t)=\langle x^i,t|\psi\rangle$ in the commutative space the standard explanations. Especially if the potential does not depend on time explicitly, one can solve the energy function

$$H\left(\hat{x}^i,\frac{\hbar}{i}\frac{\partial}{\partial x^i}\right)\psi_n(x^i)=E_n\psi_n(x^i)\,, \tag{7}$$

and then get the spectra and wave functions.

In this paper, we shall analyze this problem using the path integral formulation. In this section, we shall perform the path integral process in commutative space (4). In the next section, we shall perform the process in the momentum space.

The classical Hamiltonian of the noncommutative 2-dimensional harmonic oscillator with mass M and frequency ω_0 is

$$H=\frac{P^2}{2M}+\frac{1}{2}M\omega_0^2X^2\,, \tag{8}$$

in which the variables X^i, P_i satisfy the following classical Poisson brackets:

$$\begin{aligned} &\{X^i,X^j\}=\theta\epsilon^{ij}\,, \quad \{X^i,P_j\}=\delta^i_j\,,\\ &\{P_i,P_j\}=0\,, \quad i,j=1,2. \end{aligned} \tag{9}$$

The corresponding quantum version of the above algebras are nothing but (1) and (2).

The nonvanishing brackets for the variables X^i mean that one cannot define the propagator by $\langle \mathbf{R}_b,t_b|\mathbf{R}_a,t_a\rangle$ (in which $\mathbf{R}=X^i$, $i=1,2$) in the noncommutative plane. Nevertheless, once we map the classical noncommutative variables X^i,P_i to the classical commutative ones x^i,p_i, which are defined as

$$\{x^i,x^j\}=0\,, \quad \{x^i,p_j\}=\delta^i_j\,, \quad \{p_i,p_j\}=0\,, \tag{10}$$

using the following substitutions:

$$X^i=x^i-\frac{\theta}{2}\epsilon^{ij}p_j\,, \quad P_i=p_i\,, \tag{11}$$

we can define the propagator in the commutative plane as usual.

In terms of the commutative variables x^i,p_i, the classical Hamiltonian (8) is expressed as

$$H=\frac{1}{2M^*}p_i^2+\frac{M^*\Omega^2}{2}x_i^2-\omega\epsilon^{ij}x_ip_j\,, \tag{12}$$

in which

$$M^*=\frac{M}{1+\frac{M^2\omega_0^2\theta^2}{4}} \tag{13}$$

and

$$\Omega^2=\left(1+\frac{M^2\omega_0^2\theta^2}{4}\right)\omega_0^2\,, \quad \omega=\frac{1}{2}M^*\theta\Omega^2\,. \tag{14}$$

The canonical quantization procedure is complete, provided the above canonical variables are replaced by the corresponding operators and the classical Poisson brackets are replaced by quantum brackets, $\{,\}\to\frac{1}{\mathrm{i}\hbar}[,]$. This procedure is just the one that has been used widely. Here we shall analyze this model using the path integral formulation.

The starting point of the path integral formulation is to construct the propagator defined by [19, 20]

$$\begin{aligned} &K(\mathbf{r}_b,t_b;\mathbf{r}_a,t_a)\equiv\langle\mathbf{r}_b,t_b|\mathbf{r}_a,t_a\rangle\\ &=\int\prod_{i=1}^{2}\mathcal{D}x^i\mathcal{D}p_i\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}\mathrm{d}t\left[p_i\dot{x}^i-H\left(x^i,p_i\right)\right]\right\}, \end{aligned} \tag{15}$$

where $\mathbf{r}=x^i$, $i=1,2$ and H is the Hamiltonian. For our model, the Hamiltonian has been expressed in terms of standard canonical variables (12).

Since the integrand is the exponential of a quadratic form in the variables p_i, we can integrate them directly. Integrating with respect to the variables p_i, we arrive at

$$K(\mathbf{r}_b,t_b;\mathbf{r}_a,t_a)=N\int\prod_{i=1}^{2}\mathcal{D}x^i\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b}\mathrm{d}tL(x^i,\dot{x}^i)\right\}, \tag{16}$$

where N is a numerical factor and $L=L(x^i,\dot{x}^i)$ is the Lagrangian corresponding to the Hamiltonian (12),

$$L(x^i,\dot{x}^i)=\frac{1}{2}M^*\dot{x}_i^2+M^*\omega\epsilon^{ij}x_i\dot{x}_j-\frac{1}{2}M^*\omega_0^2x_i^2\,. \tag{17}$$

Obviously, the noncommutative parameter θ acts as a magnetic field perpendicular to the x^i plane.

Since

$$K(\mathbf{r}_b, t_b; \mathbf{r}_a, t_a) = \langle \mathbf{r}_b | \exp\left(-\mathrm{i}\frac{\hat{H}}{\hbar}T\right) |\mathbf{r}_a\rangle\,, \tag{18}$$

where $T = t_b - t_a$ and

$$\sum_n |n\rangle\langle n| = 1\,, \tag{19}$$

in which $|n\rangle$ are eigenstates of the Hamiltonian,

$$\hat{H}|n\rangle = E_n|n\rangle\,. \tag{20}$$

Sandwiching (19) into (18), we are lead to

$$\begin{aligned} K(\mathbf{r}_b, t_b; \mathbf{r}_a, t_a) &= \sum_n \exp\left(-\mathrm{i}\frac{E_n}{\hbar}T\right) \langle \mathbf{r}_b|n\rangle\langle n|\mathbf{r}_a\rangle \\ &= \sum_n \exp\left(-\mathrm{i}\frac{E_n}{\hbar}T\right) \psi_n(\mathbf{r}_b)\psi_n^*(\mathbf{r}_a)\,. \end{aligned} \tag{21}$$

So, one can read both the eigenvalues and wave functions from the propagator, once it is constructed.

There are many methods to calculate the propagator (16). We sketch only one of them.

Let x^i_{cl} be the classical path between the two specified end points $(t_a, \mathbf{r}_a)$ and $(t_b, \mathbf{r}_b)$. This is the path which is an extremum for the action $S = \int_{t_a}^{t_b} \mathrm{d}tL$.

We represent x^i in terms of x^i_{cl} and the deviations δx^i:

$$x^i = x^i_{\mathrm{cl}} + \delta x^i\,. \tag{22}$$

In fact, x^i_{cl} are the solutions of the classical equations of motion, which are obtained by minimizing the Lagrangian (17):

$$M^*\ddot{x}^i_{\mathrm{cl}} - 2M^*\omega\epsilon^{ij}\dot{x}^j_{\mathrm{cl}} + M^*\omega_0^2 x^i_{\mathrm{cl}} = 0\,, \tag{23}$$

with the ‘boundary’ conditions

$$\begin{aligned} t = t_a\,,\quad x^i = x^i_a\,,\\ t = t_b\,,\quad x^i = x^i_b\,. \end{aligned} \tag{24}$$

It is straightforward to get the solutions of the equations of motion (23) with the boundary conditions (24). After some calculations, we get

$$\begin{aligned} z_{\mathrm{cl}} &= \mathrm{e}^{-\mathrm{i}\omega(t-t_a)}\Bigg[z_a\cos\Omega(t-t_a) + \frac{z_b\mathrm{e}^{\mathrm{i}\omega T}}{\sin\Omega T}\sin\Omega(t-t_a) \\ &\qquad - z_a\cot\Omega T\sin\Omega(t-t_a)\Bigg]\,, \\ \bar{z}_{\mathrm{cl}} &= \mathrm{e}^{\mathrm{i}\omega(t-t_a)}\Bigg[\bar{z}_a\cos\Omega(t-t_a) + \frac{\bar{z}_b\mathrm{e}^{-\mathrm{i}\omega T}}{\sin\Omega T}\sin\Omega(t-t_a) \\ &\qquad - \bar{z}_a\cot\Omega T\sin\Omega(t-t_a)\Bigg]\,, \end{aligned} \tag{25}$$

where $z = x^1 + \mathrm{i}x^2$, $\bar{z} = x^1 - \mathrm{i}x^2$. Also, $z_a = x^1_a + \mathrm{i}x^2_a$, $\bar{z}_a = x^1_a - \mathrm{i}x^2_a$, as we have for z_b and $\bar{z}_b$.

Denote S_{cl} as the action along the classical trajectory,

$$\begin{aligned} S_{\mathrm{cl}} &= \int_{t_a}^{t_b} \mathrm{d}tL(x^i_{\mathrm{cl}}, \dot{x}^i_{\mathrm{cl}}) \\ &= \int_{t_a}^{t_b} \mathrm{d}t\frac{1}{2}M^*\dot{x}^{i2}_{\mathrm{cl}} + M^*\omega\epsilon^{ij}x^i_{\mathrm{cl}}\dot{x}^j_{\mathrm{cl}} - \frac{1}{2}M^*\omega_0^2 x^{i2}_{\mathrm{cl}}\,. \end{aligned} \tag{26}$$

Taking account of the classical equations of motion, we can write the classical action as

$$S_{\mathrm{cl}} = \frac{1}{4}M(z\dot{\bar{z}} + \bar{z}\dot{z})|_{t_a}^{t_b}\,. \tag{27}$$

The explicit expression for the classical action can be obtained by substituting (25) into (27),

$$\begin{aligned} S_{\mathrm{cl}} = -\frac{M^*\Omega}{2\sin\Omega T}\Big[&\left(z_a\bar{z}_b\mathrm{e}^{-\mathrm{i}\omega T} + \bar{z}_a z_b\mathrm{e}^{\mathrm{i}\omega T}\right) \\ &- \cos\Omega T(z_a\bar{z}_a + z_b\bar{z}_b)\Big]\,. \end{aligned} \tag{28}$$

Substituting (22) into (17), we can write the propagator as

$$K(\mathbf{r}_b, t_b; \mathbf{r}_a, t_a) = N\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\mathrm{cl}}}F(t_a, t_b)\,, \tag{29}$$

where S_{cl} has been given in (28), and $F(t_a, t_b)$ is the prefactor (or quantum fluctuation).

Since initial and final points are fixed at $\mathbf{r}_a, \mathbf{r}_b$ respectively, the deviations vanish at the end points:

$$\delta x^i(t_a) = \delta x^i(t_b) = 0\,,\quad i = 1, 2\,. \tag{30}$$

So the prefactor $F(t_a, t_b)$ can be written as

$$F(t_a, t_b) = \int_0^0 \prod_{i=1}^2 \mathcal{D}(\delta x^i)\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}tL(\delta x^i, \delta\dot{x}^i)\right]\,. \tag{31}$$

Furthermore, since the Lagrangian (17) is invariant under time translation, the prefactor only depends on the time difference $T = t_b - t_a$. As a result, the prefactor can be written as

$$\begin{aligned} F(t_a, t_b) &= F(T) \\ &= \int_0^0 \prod_{i=1}^2 \mathcal{D}(\delta x^i)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}tL(\delta x^i, \delta\dot{x}^i)\right\}\,, \end{aligned} \tag{32}$$

in which

$$\begin{aligned} L(\delta x^i, \delta\dot{x}^i) = \frac{1}{2}M^*(\delta\dot{x}_i)^2 + M^*\omega\epsilon^{ij}\delta x_i\delta\dot{x}_j \\ - \frac{1}{2}M^*\omega_0^2(\delta x_i)^2\,. \end{aligned} \tag{33}$$

Integrating by parts and noticing the boundary conditions for the deviation δx^i in (30), we can write the prefactor (32) as

$$F(T)=\int_0^0\prod_{i=1}^{2}\mathcal{D}(\delta x^i)\exp\{\frac{\mathrm{i}M^*}{2\hbar}\int_{t_a}^{t_b}\mathrm{d}t\delta x^i\Delta_{ij}\delta x^j\}\,,\tag{34}$$

where Δ_{ij} is an operator defined by

$$\Delta_{ij}=-\delta_{ij}(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\omega_0^2)+2\omega\epsilon_{ij}\frac{\mathrm{d}}{\mathrm{d}t}\,.\tag{35}$$

Obviously, the integrand is an exponential of the quadratic form (Gaussian type), and we can integrate it as follows:

$$F(T)\sim[\det\Delta_{ij}]^{-\frac{1}{2}}\sim\prod_{n=1}^{\infty}\left(\frac{n^2\pi^2}{T^2}\right)^{-1}\prod_{n=1}^{\infty}\left[1-\frac{\Omega^2T^2}{n^2\pi^2}\right]^{-1}.\tag{36}$$

Since the first product does not depend on Ω, we collect it and other factors into a single constant. As for the second factor, we use the well-known formula

$$\sin x=x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{n^2\pi^2}\right).\tag{37}$$

So the prefactor $F(T)$ is expressed as

$$F(T)=C\frac{\Omega T}{\sin\Omega T}\,,\tag{38}$$

where C is a constant that is independent of Ω. It can be uniquely determined by noticing that the integral is a 2-dimensional harmonic oscillator when $\omega\to 0$ or a charged particle moving in a plane with a constant perpendicular magnetic field in the symmetric gauge when $\omega_0\to 0$. The final result is

$$K(\mathbf{r}_b,t_b;\mathbf{r}_a,t_a)=\frac{M^*\Omega}{2\pi\mathrm{i}\hbar\sin\Omega T}\times\exp\left\{-\frac{\mathrm{i}M^*\Omega}{2\hbar\sin\Omega T}\left[\left(z_a\bar z_b\mathrm{e}^{-\mathrm{i}\omega T}+\mathrm{c.c.}\right)-\cos\Omega T(z_a\bar z_a+z_b\bar z_b)\right]\right\},\tag{39}$$

in which c.c. means the complex conjugate.

The eigenvalues and wave functions can be read from the above formula (an analogous process can be found in [21]; here we only list the result). They are

$$E_{n_r,m}=2\hbar\Omega\left(n_r+\frac{1}{2}+\frac{|m|}{2}-\frac{m\omega}{2\Omega}\right),\tag{40}$$
$$n_r=0,1,2,\cdots,\quad m=0,\pm1,\pm2,\cdots$$

and

$$\psi_{n_r,m}(\mathbf{r})=\frac{1}{\sqrt{2\pi r}}R_{n_r,m}(r)\mathrm{e}^{\mathrm{i}m\theta}\,,\tag{41}$$

in which $R_{n_r,m}(r)$ is the radial wave function

$$R_{n_r,m}=\sqrt{r}\left(\frac{2M^*\Omega}{\hbar}\right)^{\frac{1}{2}}\sqrt{\frac{n_r!}{(n_r+|m|)!}}\exp\left(-\frac{M^*\Omega}{2\hbar}r^2\right)\times\left(\frac{M^*\Omega}{\hbar}r^2\right)^{|m|/2}L_{n_r}^{|m|}\left(\frac{M^*\Omega}{\hbar}r^2\right)\tag{42}$$

and $L_{n_r}^{|m|}$ is for the Laguerre polynomials.

3 Path integral in momentum space

In the above section, we studied the noncommutative 2-dimensional harmonic oscillator by employing the path integral formulation. Because of the noncommutativity of the coordinates in noncommutative space, we cannot define the propagator in the noncommutative space directly. However, we notice that the commutators among the momenta are commutative. As a result, the wave functions in the momentum space $\varphi(\mathbf{P},t)=\langle\mathbf{P},t|\varphi\rangle$ still have their exact meanings. So it is possible to calculate the propagator in momentum space and then read both the eigenvalues and wave functions from it.

According to the definition [19–21], the propagator in momentum space is defined by

$$K(\mathbf{P}_b,t_b;\mathbf{P}_a,t_a)\equiv\langle\mathbf{P}_b,t_b|\mathbf{P}_a,t_a\rangle\,.\tag{43}$$

Considering (19), we can rewrite the propagator (43) as

$$K(\mathbf{P}_b,t_b;\mathbf{P}_a,t_a)=\langle\mathbf{P}_b|\exp\left\{-\mathrm{i}\frac{\hat H}{\hbar}T\right\}|\mathbf{P}_a\rangle=\sum_n\exp\left(-\mathrm{i}\frac{E_n}{\hbar}T\right)\varphi_n(\mathbf{P}_b)\varphi_n^*(\mathbf{P}_a)\,.\tag{44}$$

One can, of course, read both eigenvalues and wave functions in momentum space from it once it is determined.

We go on in the standard way writing the propagator (44) in the form [21]

$$K(\mathbf{P}_b,t_b;\mathbf{P}_a,t_a)=\int\prod_{i=1}^{2}\mathcal{D}X^i\mathcal{D}P_i\exp\left[\frac{\mathrm{i}}{\hbar}S_\mathrm{p}\right],\tag{45}$$

where $S_\mathrm{p}=\int_{t_a}^{t_b}\mathrm{d}tL_\mathrm{p}$ is the action in the phase space (which is spanned by the variables X^i and their canonical conjugate momenta P_i) we shall construct.

There are two principles one must notice when the Lagrangian L_p is constructed. One is that the Lagrangian should lead to the commutator (1) and (2); the other is that it should be consistent with the Hamiltonian (8). It can be verified by employing Faddeev–Jackiw theory [22] that the following first-order Lagrangian satisfies the above principles:

$$L_\mathrm{p}=\frac{1}{2}\left(P_i\dot X^i-X^i\dot P_i\right)+\frac{1}{2}\theta\epsilon^{ij}P_i\dot P_j-H\,,\tag{46}$$

where H is the Hamiltonian that has been given in (8).

The explicit expression of the action S_{p} is

$$S_{\mathrm{p}} = \int_{t_a}^{t_b} \mathrm{d}t L_{\mathrm{p}} = \int_{t_a}^{t_b} \mathrm{d}t \frac{1}{2}\left(P_i \dot{X}^i - X^i \dot{P}_i\right) + \frac{1}{2}\theta\epsilon^{ij} P_i \dot{P}_j - H\,. \tag{47}$$

Integrating by parts and dropping the surface term, we can rewrite the above action as

$$S_{\mathrm{p}} = \int_{t_a}^{t_b} \mathrm{d}t - X^i \dot{P}_i + \frac{1}{2}\theta\epsilon^{ij} P_i \dot{P}_j - H\,. \tag{48}$$

Since the integrand (45) is an exponential of a quadratic form in the variables X^i, we integrate them directly. The result is

$$K(\mathbf{P}_b, t_b; \mathbf{P}_a, t_a) = N \int \prod_{i=1}^{2} \mathcal{D}P_i \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t_a}^{t_b} \mathrm{d}t L_{\mathrm{m}}(P_i, \dot{P}_i)\right], \tag{49}$$

where N is a numerical factor and $L_{\mathrm{m}}(P_i, \dot{P}_i)$ is the Lagrangian in momentum space,

$$L_{\mathrm{m}} = \frac{1}{M^2\omega_0^2}\left(\frac{1}{2}M\dot{P}_i^2 + M\omega\epsilon_{ij}P_i\dot{P}_j - \frac{1}{2}M\omega_0^2 P_i^2\right), \tag{50}$$

where the parameter ω has been introduced in (14).

The Lagrangian (50) is dual to the one that describes a charged harmonic oscillator moving in a plane with a perpendicular constant magnetic field in coordinate space. Clearly, the noncommutative parameter θ serves again as a magnetic field.

We can follow the same scheme to calculate the propagator in the momentum space corresponding to the Lagrangian (50). Here we only list the result:

$$K(\mathbf{P}_b, t_b; \mathbf{P}_a, t_a) = \frac{2\pi\hbar\Omega}{\mathrm{i}M\omega_0^2 \sin\Omega T} \exp\left\{-\frac{\mathrm{i}\Omega}{2M\hbar\omega_0^2 \sin\Omega T}\left[\left(P_a\bar{P}_b \mathrm{e}^{-\mathrm{i}\omega T} + \mathrm{c.c}\right) - \cos\Omega T (P_a\bar{P}_a + P_b\bar{P}_b)\right]\right\}, \tag{51}$$

where $P_a = P_a^1 + P_a^2$, etc. Ω has been given in (14).

From the above propagator, one can read both eigenvalues and wave functions from it.

The expression of the eigenvalues is

$$E_{n_r,m} = 2\hbar\Omega\left(n_r + \frac{1}{2} + \frac{|m|}{2} - \frac{m\omega}{2\Omega}\right), \quad n_r = 0,1,2,\cdots, \quad m = 0,\pm1,\pm2,\cdots, \tag{52}$$

which coincide with the ones obtained in the coordinate path integral formulation (40). The expression for the wave functions is

$$\varphi_{n_r,m}(\mathbf{P}) = \frac{1}{\sqrt{2\pi P}} R_{n_r,m}(P)\mathrm{e}^{\mathrm{i}m\theta}, \tag{53}$$

in which $P = |\mathbf{P}|$ and $R_{n_r,m}(P)$ are radial wave functions in momentum space,

$$R_{n_r,m}(P) = \sqrt{P}\left(\frac{2\Omega}{M\hbar\omega_0^2}\right)^{\frac{1}{2}} \sqrt{\frac{n_r!}{(n_r+|m|)!}} \times \exp\left(-\frac{\Omega P^2}{2M\hbar\omega_0^2}\right)\left(\frac{\Omega P^2}{M\hbar\omega_0^2}\right)^{|m|/2} \times L_{n_r}^{|m|}\left(\frac{\Omega P^2}{M\hbar\omega_0^2}\right). \tag{54}$$

4 Conclusions and remarks

In the previous sections, we get the eigenvalues and wave functions of the 2-dimensional harmonic oscillator from the propagators (39) and (51) both in coordinate space and momentum space, respectively.

For the former, because of the noncommutative relations among the coordinates (1), the wave functions in noncommutative space lose their exact meaning. As a result, one cannot define the propagator in noncommutative coordinate space directly. In order to employ the path integral formulation, we map the noncommutative plane to the commutative one and then calculate the propagator in this commutative plane. Since the integrand is an exponential of a Gaussian form in the momenta, we integrate them and get the Lagrangian (17). This Lagrangian is analogous to the one that describes a charged harmonic oscillator moving in a plane with a uniform perpendicular magnetic field. We find that one of the most important effects of the noncommutative parameter θ is that it serves as a perpendicular magnetic field.

For the latter, since the momenta are commutative, the wave functions in momentum space still have meaning. We can perform the path integral formulation directly in the momentum space. And since the integrand is a Gaussian form in the coordinate, we can integrate them and get the Lagrangian (50) in momentum space. This Lagrangian is dual to the one that describes a charged harmonic oscillator moving in a plane with a perpendicular constant magnetic field in coordinate space. Interestingly, the noncommutative parameter also serves as the perpendicular magnetic field.

We are interested in the behaviors of the eigenvalues and wave functions when the noncommutative parameter θ takes certain limits. First, let us consider the limit of $\theta \to 0$.

Notice that

$$\lim_{\theta\to0}\omega = 0\,, \quad \lim_{\theta\to0}\Omega = \omega_0\,. \tag{55}$$

It means that when the noncommutative parameter θ takes its zero limit, the spectra will converge smoothly to

$$E_{n_r,m} = 2\hbar\omega_0\left(n_r + \frac{1}{2} + \frac{|m|}{2}\right) \quad n_r = 0,1,2,\cdots, \quad m = 0,\pm1,\pm2,\cdots \tag{56}$$

It is the spectrum of a 2-dimensional harmonic oscillator in the commutative plane. The corresponding radial wave functions in this limit can also be obtained. They are

$$R_{n_r,m}(r) = \sqrt{r}\left(\frac{2M\omega_0}{\hbar}\right)^{\frac{1}{2}}\sqrt{\frac{n_r!}{(n_r+|m|)!}}\exp\left(-\frac{M\omega_0}{2\hbar}r^2\right) \times\left(\frac{M\omega_0}{\hbar}r^2\right)^{|m|/2}L_{n_r}^{|m|}\left(\frac{M\omega_0}{\hbar}r^2\right), \quad (57)$$

in coordinate space and

$$R_{n_r,m}(P) = \sqrt{p}\left(\frac{2}{M\hbar\omega_0}\right)^{\frac{1}{2}}\sqrt{\frac{n_r!}{(n_r+|m|)!}}\exp\left(-\frac{P^2}{2M\hbar\omega_0}\right) \times\left(\frac{P^2}{M\hbar\omega_0}\right)^{|m|/2}L_{n_r}^{|m|}\left(\frac{P^2}{M\hbar\omega_0}\right), \quad (58)$$

in momentum space, respectively.

Then, we consider the large θ limit. In [15–17] the authors find a perturbative expansion of the energy level for NCQM with a central potential. In our model, we can find the exact expression.

In coordinate space, the following expressions are valid if the noncommutative parameter is large:

$$M^* = \frac{4}{M\theta^2\omega_0^2}, \quad \Omega = \omega = \frac{1}{2}M\theta\omega_0^2. \quad (59)$$

These mean that the spectra are

$$E_{n_r,m} = 2\hbar\omega\left(n_r + \frac{|m|}{2} - \frac{m}{2} + \frac{1}{2}\right). \quad (60)$$

The same expression for the energy levels can be found in momentum space in the large θ limit.

In fact, (60) is analogous to Landau levels with θ acting as a constant perpendicular magnetic field. Obviously, only the lowest level is possible in this limit.

Acknowledgements. The authors thank Dr. Bin Zhou for his valuable discussions. This work is supported by NSF of China (Grant No. 10505003) and the Program for New Century Excellent Talents in University as well as the Outstanding Youth Foundation of GuiZhou province (Grant No. 20050530).

References

1. A. Connes, M. Douglas, A.S. Schwarz, JHEP **9802**, 003 (1998)
2. N. Seiberg, E. Witten, JHEP **99009**, 032 (1999)
3. M. R Douglas, N.A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001)
4. C.S. Chu, P.M. Ho, Nucl. Phys. B **550**, 151 (1999)
5. C.S. Chu, P.M. Ho, Nucl. Phys. B **568**, 447 (2000)
6. F. Ardalan, H. Arfaei, M.M. Sheikh-Jabbari, Nucl. Phys. B **576**, 578 (2000)
7. J. Jing, Z.-W. Long, Phys. Rev. D **72**, 126 002 (2005)
8. S. Minwalla, M. Van Raamsdonk, N. Seiberg, JHEP **0002**, 020 (2000)
9. M. Van Raamsdonk, N. Seiberg, JHEP **0003**, 035 (2000)
10. R. Gopakumar, S. Minwalla, A. Strominger, JHEP **0005**, 020 (2000)
11. V.P. Nair, A.P. Polychronakos, Phys. Lett. B **505**, 267 (2001)
12. B. Morariu, A.P. Polychronakos, Nucl. Phys. B **610**, 531 (2001)
13. D. Karabali, V.P. Nair, A.P. Polychronakos, Nucl. Phys. B **627**, 565 (2002)
14. B. Morariu, A.P. Polychronakos, Nucl. Phys. B **634**, 326 (2002)
15. J. Gamboa, M. Loewe, J.C. Rojas, Phys. Rev. D **64**, 067 901 (2001)
16. J. Gamboa, M. Loewe, F. Mendez, J.C. Rojas, Mod. Phys. Lett. A **16**, 2075 (2001)
17. J. Gamboa, M. Loewe, F. Mendez, J.C. Rojas, Int. J. Mod. Phys. A **17**, 2555 (2002)
18. B. Muthukumar, P. Mitra, Phys. Rev. D **66**, 027 701 (2002)
19. R.P. Feynman, Rev. Mod. Phys. **20**, 367 (1948)
20. R.P. Feynman, A.R. Hibbs, Quantum Mechanics and Path Integrals (McGrawHill, New York, 1965)
21. H. Kleinert, Path Integral in Quantum Mechanics, Statistics and Polymer Science, 3nd edn. (World Sciengific, Singapore, 2004)
22. L.D. Faddeev, R. Jackiw, Phys. Rev. Lett. **60**, 1692 (1988)